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Addition and Subtraction of Ideals

Michael Maltenfort

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Let A be a Noetherian ring, $\tilde{I}$ and I be comaximal ideals of A, and P be a projective A-module. "Addition" refers to being given a surjection $P \twoheadrightarrow \tilde{I}$ and producing a surjection $P \twoheadrightarrow \tilde{I} \cap I$. This is useful, for example, when P is free, to determine how many elements it takes to generate the ideal $\tilde{I} \cap I$. "Subtraction" refers to being given $P \twoheadrightarrow \tilde{I} \cap I$ and producing some $P \twoheadrightarrow \tilde{I}$. A major use of this is when $\tilde{I} = A$, to show a projective module has a unimodular element. Here we extend certain addition and subtraction results of R. Sridharan (1995, *J. Algebra* **176,** 947–958) and S. Mandal and R. Sridharan (1996, *J. Math. Kyoto Univ.* **36,** No. 3, 453–470). In both addition and subtraction, we weaken the hypotheses imposed on $\tilde{I}$ and in a suitable fashion remove the hypothesis that P must have trivial determinant. With certain restrictions, we also now allow $\dim A/I \le 1$, rather than $\operatorname{ht} I = \dim A$. When A is an affine algebra over a field F, Mandal and Sridharan had subtraction results for when I is the intersection of finitely many maximal ideals whose residue fields are quadratically closed, when $\sqrt{I}$ has this form (if char $F \neq 2$), or when I is the maximal ideal of an F-rational point. In our generalization, we allow I to be the intersection of finitely many maximal ideals, all of whose residue fields are quadratically closed, except possibly one which instead defines an F-rational point. When char $F \neq 2$, we only need to require $\sqrt{I}$ to have this form. In particular, we can subtract an ideal whose radical is the maximal ideal of an F-rational point; this has been used by S. M. Bhatwadekar and R. Sridharan (1998, *Invent. Math.* **133,** No. 1, 161–192) to construct a certain local complete intersection ideal which is not a complete intersection ideal.

0. NOTATIONS, ASSUMPTIONS, AND TERMINOLOGY

In this paper, we assume that all rings are commutative with unit. Let A be a ring. As usual, $\dim A$ will be Krull dimension, which by convention will be $-\infty$ for the zero ring, so that if $I = A$, then $\dim A/I \le k$ for any $k \in \mathbb{Z}$. We denote by $A^\times$ the group of units of A, and by $A[t]$ the polynomial ring in one variable over A. For an A-module M, $M[t]$

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denotes $M \otimes_A A[t]$. For $a \in A$ and $I \subset A[t]$, $I(a)$ is the image of I under the map $A[t] \to A$ which sends $t \mapsto a$. We say ideals $\tilde{I}$, $I \subseteq A$ are comaximal if $\tilde{I} + I = A$.

We say an ideal $I \subseteq A$ is a complete intersection ideal of height r if it is generated by a regular sequence of length r. It is a local complete intersection ideal of height r if for all primes $\mathfrak{p} \in \operatorname{Spec} A$ which contain I, $I_\mathfrak{p}$ is a complete intersection ideal of height r. It is a standard fact that a local complete intersection of height r is a complete intersection of height r if and only if it is generated by r elements.

We assume all projective modules are finitely generated, but not necessarily of constant rank. For P a projective A-module, "$r = \operatorname{rank} P$," means that r is the function $\operatorname{Spec} A \to \mathbb{N}$ given by $\mathfrak{p} \mapsto \operatorname{rank}_{A_\mathfrak{p}} P_\mathfrak{p}$. So for $n \in \mathbb{N}$, "$r \geq n$" means that $r(\mathfrak{p}) \geq n$, for all $\mathfrak{p} \in \operatorname{Spec} A$, i.e., we think of n as a constant function on $\operatorname{Spec} A$. And $\wedge^r P$ is the rank 1 projective A-module, sometimes called the determinant of P, which equals the top exterior power of P on each open subset of $\operatorname{Spec} A$ on which P has constant rank. (Note, however, that little is lost by assuming that all projective modules have constant rank.) For a ring A and P a projective A-module of rank r, we denote by $\operatorname{End}(P)$ the set of A-module endomorphisms of P, $\operatorname{GL}(P)$ the group of A-module automorphisms, and by $\operatorname{SL}(P)$ the subgroup of automorphisms α with $\wedge^r \alpha = 1$.

For M an A-module, we write M^* for $\operatorname{Hom}_A(M, A)$. If $m \in M$, we say m is a unimodular element if there exists $f \in M^*$ such that $f(m) = 1$, or equivalently, m generates a rank 1 free direct summand of M. We denote by 1_M the identity map on M. For $n \geq 1$, we denote by 1_n the $n \times n$ identity matrix. For F a field, an affine algebra over F is an F-algebra A of finite type. An F-rational point of A is a closed point of $\operatorname{Spec} A$ defined by a maximal ideal $\mathfrak{m} \subseteq A$ such that $A/\mathfrak{m}$ is isomorphic to F as an F-algebra.

1. INTRODUCTION

Let A be a Noetherian ring, $\tilde{I}$ and I be comaximal ideals of A, and P be a projective A-module. Following the terminology of [MS], "addition" refers to being given a surjection $P \twoheadrightarrow \tilde{I}$ and producing a surjection $P \twoheadrightarrow \tilde{I} \cap I$. This is useful, for example, when P is free, to determine how many elements it takes to generate the ideal $\tilde{I} \cap I$. "Subtraction" refers to being given $P \twoheadrightarrow \tilde{I} \cap I$ and producing some $P \twoheadrightarrow \tilde{I}$. A major use of this is when $\tilde{I} = A$, to show a projective module has a unimodular element. In this paper we extend certain addition and subtraction results of Sridharan [RS1] and Mandal and Sridharan [MS].

One motivation for the search for unimodular elements is the following well known situation. (See [Mu, MK2, MKMu].) Suppose A is a smooth affine variety of dimension n over an algebraically closed field, P is a rank n projective A-module, and J is the image of a generic section of P. Then P having a unimodular element is equivalent to $0 = c_n(P)$ (the nth chern class), and also equivalent to J being a complete intersection ideal. This equivalence, however, does not hold if the base field is not algebraically closed. Nori suggested an "euler class map" to replace the nth chern class map, so that P has a unimodular element if and only if its euler class is zero. This has been carried out for P having trivial determinant in [MS, RS3, and BS]. Our subtraction results allow us to expand the class of ideals J such that the existence of $P \twoheadrightarrow J$ implies P has a unimodular element. We also will allow P to have non-trivial determinant, which may lead to a more general "euler class map" which would be able to determine the existence of unimodular elements for such P. Note that in all our cases of subtraction, one of our addition theorems tells us that in fact P having a unimodular element is equivalent to the existence of some $P \twoheadrightarrow J$.

Consider the following theorem of Mohan Kumar.

THEOREM [MK2, Corollary 1]. *Let A be a reduced affine algebra of dimension n over a field which is finite or algebraically closed. Suppose $\tilde{I}$ and I are comaximal ideals which are local complete intersection ideals of height n. If $\tilde{I}$ and I are generated by n elements, then $\tilde{I} \cap I$ is generated by n elements.*

Sridharan extended the above result (except when $n \le 2$):

THEOREM [RS1, Theorem 4]. *Let A be a Noetherian ring of dimension n. Suppose $\tilde{I}$ and I are comaximal ideals of height r which are generated by r elements. Further suppose $2r \ge n + 3$ and $\dim A/\tilde{I} = \dim A/I = n - r$. Then $\tilde{I} \cap I$ is generated by r elements.*

The above theorem is a special case (with $Q = A^n$) of (1) in the following theorem, which summarizes Theorem 3.5 and Theorem 3.6.

THEOREM. *Let A be a Noetherian ring of dimension n. Let P and Q be rank r projective A-modules, Q having a unimodular element. Suppose $\tilde{I}$ and I are comaximal ideals of A with $\dim A/\tilde{I} \le r - 2$. If there exist surjections $Q \twoheadrightarrow I$ and $P \twoheadrightarrow \tilde{I}$, then there exists a surjection $P \twoheadrightarrow \tilde{I} \cap I$, provided we have one of the following sets of conditions.*

(1) $P = Q$, $\dim A/I \le r - 2$, *and* $2r \ge n + 3$.

(2) $\dim A/I = 0$, *and* $r \ge n + 1$.

(3) $\dim A/I \le 1$, $r \ge \max\{n, 3\}$, *and* $\wedge^r P \cong \wedge^r Q$.

Note that by taking $Q = A^n$ in (3), we get a more general statement of the following addition principle of Mandal and Sridharan.

THEOREM [MS, Theorem 3.2]. *Let A be a Noetherian ring of dimension $n \geq 3$ and P be a projective A-module of rank n with trivial determinant. Let $\tilde{I}$ and I be comaximal ideals of height n. If there exist surjections $A^n \twoheadrightarrow I$ and $P \twoheadrightarrow \tilde{I}$, then there exists a surjection $P \twoheadrightarrow \tilde{I} \cap I$.*

Sridharan also proved the following result [RS1, Theorem 1]. Let $\dim A = n \geq 3$, $I \subseteq A$ be a height n ideal generated by n elements, and P be a rank n projective module with trivial determinant. If P has a unimodular element, then there exists a surjection $P \twoheadrightarrow I$. Sridharan's proof uses Mandal's Theorem [Ma, Theorem 2.1]. Mohan Kumar pointed out (oral communication) that it suffices to use only the Eisenbud−Evans Theorem, i.e., Theorem 2.2, below. This allows us to prove the following result, which we will prove as Corollary 3.3. (Sridharan's result is the special case of (2) with $Q = A^n$ and $r = \operatorname{ht} I = n \geq 3$.)

THEOREM. *Let A be a Noetherian ring of dimension n, $I \subseteq A$ an ideal, and P and Q projective A-modules of rank r, each having a unimodular element. Suppose we have one of the following conditions.*

(1) $\dim A/I = 0$ *and* $r \geq n + 1$.

(2) $\dim A/I \leq 1$, $r \geq n$, *and* $\wedge^r P \cong \wedge^r Q$.

If there exists a surjection $Q \twoheadrightarrow I$, then there exists a surjection $P \twoheadrightarrow I$.

The above result can be interpreted to say that, with certain restrictions, the existence of a surjection $P \twoheadrightarrow I$ depends only on rank P and possibly its determinant. We can also think of this as an addition result, taking $P \twoheadrightarrow A$ to $P \twoheadrightarrow I$. Although the above theorem is barely an improvement over the addition theorem stated above (only the case $r = n = 2$ is new), here we will have more control over the surjection we find. For a more precise statement, see Theorem 3.2.

Now let us consider subtraction. In Theorem 4.1, we will see that any ideal can be subtracted when rank $P \geq 1 + \dim A$. Our main result, Theorem 6.1, addresses the case rank $P = \dim A$. The following theorem gives one consequence of this theorem.

THEOREM. *Let A be an affine algebra of dimension $n \geq 3$ over a field F with* $\operatorname{char} F \neq 2$. *Let P be a projective A-module with trivial determinant, and $I \subseteq A$ an ideal generated by n elements such that $\sqrt{I}$ defines an F-rational point. If there exists a surjection $P \twoheadrightarrow I$, then P has a unimodular element.*

In [RS1, MS], similar theorems were proved (see below), but with I itself the maximal ideal of an F-rational point. Our result has been used by

Bhatwedekar and Sridharan [BS] to show that the hypotheses in the following results are necessary.

THEOREM. *Let A be a regular affine domain of dimension n over a field k. Suppose $I \subseteq A[t]$ is a local complete intersection ideal of height n such that I/I^2 is generated by n elements.*

(1) [BS, *Corollary* 3.9] *If k is algebraically closed, $n \geq 3$, and $I(0)$ is either a complete intersection ideal or equal to A, then I is a complete intersection ideal.*

(2) [BS, *Corollary* 4.18] *If $k = \mathbb{R}$, $n \geq 2$, and every maximal ideal of A is a complete intersection ideal, then I is a complete intersection ideal.*

In (1), the hypothesis that k is algebraically closed is necessary, and in (2), the hypothesis that all maximal ideals are complete intersection ideals is necessary. Specifically, in [BS, Example 5.2], Bhatwadekar and Sridharan use our above subtraction result to show that if A is the coordinate ring of the real n-sphere, n even, then there exists $I \subseteq A[t]$ a local complete intersection ideal of height n which is not a complete intersection ideal, with I/I^2 generated by n elements, and $I(0)$ a complete intersection ideal.

Before stating more of our subtraction results, let us look at the subtraction results of [MS]. (These generalize [RS1, Theorem 2, 3, and 5].)

THEOREM. *Let A be a Noetherian ring of dimension $n \geq 3$ and P be a projective A-module of rank n with trivial determinant. Let $\tilde{I}$ and I be comaximal ideals of A of height n. Suppose we have surjections $A^n \twoheadrightarrow I$ and $P \twoheadrightarrow \tilde{I} \cap I$. Then there exists a surjection $P \twoheadrightarrow \tilde{I}$ provided either of the following sets of condition holds.*

(1) [MS, *Theorem* 3.5] *A is an affine algebra over a field F and I is the maximal ideal of an F-rational point.*

(2) [MS, *Theorem* 3.14] *I/I^2 is a free A/I-module of rank n and one of the following conditions holds.*

(i) *A is a finitely generated $\mathbb{Z}$-algebra.*

(ii) *A is an affine algebra over a field F and A/I is the product of quadratically closed fields (i.e., every element has a square root).*

(iii) *A is an affine algebra over a field F,* char $F \neq 2$, *and $A/\sqrt{I}$ is a product of quadratically closed fields.*

The above condition that I/I^2 is rank n free turns out to be unnecessary. Here are more consequences of our main subtraction result, Theorem 6.1. As remarked above, in these cases, our addition results tell us that in fact existence of $P \twoheadrightarrow \tilde{I} \cap I$ is equivalent to existence of $P \twoheadrightarrow \tilde{I}$.

THEOREM. *Let A be a Noetherian ring of dimension $n \geq 3$, and P and Q be projective A-modules of rank n with isomorphic determinants, Q having a unimodular element. Let $\tilde{I}$ and I be comaximal ideals with* $\dim A/\tilde{I} \leq n-2$. *If there are surjections $Q \twoheadrightarrow I$ and $P \twoheadrightarrow \tilde{I} \cap I$, then there exists a surjection $P \twoheadrightarrow \tilde{I}$, provided we have one of the following conditions.*

(1) *I is a local complete intersection ideal of height $n-1$.*

(2) *I is the finite intersection of maximal ideals of height $\leq n-1$ which define regular points. (Of course, no such I exists in an affine domain over a field.)*

(3) $\dim A/I = 0$ *and A is a finite type $\mathbb{Z}$-algebra.*

(4) *A is an affine algebra over a field F, $I = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_t$, the intersection of finitely many maximal ideals, with the residue fields for $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ [resp. $\mathfrak{m}_2, \ldots, \mathfrak{m}_t$] quadratically closed [resp. with also $\mathfrak{m}_1$ defining an F-rational point].*

(5) *A is an affine algebra over a field F of characteristic not 2, and $\sqrt{I} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_t$, the intersection of finitely many maximal ideals, with the residue fields for $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ [resp. $\mathfrak{m}_2, \ldots, \mathfrak{m}_t$] quadratically closed [resp. with also $\mathfrak{m}_1$ defining an F-rational point].*

Important special cases are $\tilde{I} = A$ (giving us a tool to find a unimodular element of P), and $Q = A^n$, i.e., I is generated by n elements, though this limits P to having trivial determinant.

In Section 2, we will state theorems of Eisenbud–Evans, Serre, and Mandal–Sridharan, and give some applications of these theorems. Section 3 has our addition results. In Section 4, we will prove our subtraction principle, which gives an abstract statement about when we can subtract. Section 6 contains our main subtraction result, Theorem 6.1, and its applications; this theorem is based on our subtraction principle and some assorted results which we prove in Section 5.

2. SOME PRELIMINARY RESULTS

We first wish to state a sufficiently general version of the Eisenbud–Evans Theorem to suit our needs.

DEFINITION. Let A be a ring, X a subset of Spec A, and $d\colon X \to \mathbb{N} = \{0, 1, \ldots\}$ any map. For $\mathfrak{p}, \mathfrak{q} \in X$, we say $\mathfrak{p} \ll \mathfrak{q}$ if $\mathfrak{p} \subsetneq \mathfrak{q}$ and $d(\mathfrak{p}) > d(\mathfrak{q})$. This gives a partial order on X. We say d is a *generalized dimension function* if for any ideal $I \subseteq A$, $V(I) \cap X$ has only a finite number of minimal elements with respect to the above partial order.

EXAMPLES 2.1.

(1) If A is Noetherian and $X \subseteq \operatorname{Spec} A$ is locally closed, then $d(\mathfrak{p}) = \dim A/\mathfrak{p}$ and $d(\mathfrak{p}) = \dim X - \operatorname{ht} \mathfrak{p}$ are generalized dimension functions.

(2) For A Noetherian and $X \subseteq \operatorname{Spec} A$ locally closed, $d(\mathfrak{p}) = r - \operatorname{ht} \mathfrak{p}$ is a generalized dimension function on $X \cap \{\mathfrak{p}: \operatorname{ht} \mathfrak{p} \le r\}$.

(3) If $d_i: X_i \to \mathbb{N}$ are generalized dimension functions for $i = 1, 2$, then so is $d: X_1 \cup X_2 \to \mathbb{N}$ given by $d(\mathfrak{p}) = \max\{d_1(\mathfrak{p}), d_2(\mathfrak{p})\}$, where we set $d_i(\mathfrak{p}) = 0$ for $\mathfrak{p} \notin X_i$.

DEFINITION. Let A be a ring, X a subset of $\operatorname{Spec} A$, Q a projective A-module, and $q \in Q$. We say q is *basic* on X if for all $\mathfrak{p} \in X$, the image of q in $Q_{\mathfrak{p}}$ is unimodular, or equivalently $\{f(q): f \in Q^*\} \not\subseteq \mathfrak{p}$. (Note that q is unimodular if and only if it is basic on $\operatorname{Spec} A$.)

THEOREM 2.2 (Eisenbud–Evans). *Let A be a ring, X a subset of* $\operatorname{Spec} A$, *and $d: X \to \mathbb{N}$ a generalized dimension function. Let Q be a projective module of rank r, and $M \subseteq Q$ be a finitely generated submodule. Suppose that for all $\mathfrak{p} \in X$, $r(\mathfrak{p}) \ge 1 + d(\mathfrak{p})$ and $M_{\mathfrak{p}} = Q_{\mathfrak{p}}$. Then for any element $(q, a) \in Q \oplus A$ which is basic on X, there exists $m \in M$ such that $q + am \in Q$ is basic on X.*

Remark 2.3. When not otherwise specified, we will apply Theorem 2.2 with $M = Q$ and $d(\mathfrak{p}) = \dim A/\mathfrak{p}$.

Sketch of Proof. When $M = Q$, this is a special case of Plumstead's extension, found in [P], of the original Eisenbud–Evans Theorem [EE, Theorem A(i)]. For the general case, use Plumstead's technique to extend [EE, Theorem A(ii)(b)] (or to extend [Sw, Lemma 3.7, Theorem 3.5]).

We will frequently use the following well known result, which follows directly from Theorem 2.2 (Eisenbud–Evans).

THEOREM 2.4 (Serre). *Let A be a Noetherian ring and P a projective A-module with* $\operatorname{rank} P \ge 1 + \dim A$. *Then P has a unimodular element*.

COROLLARY 2.5. *Let A be a Noetherian ring and P and Q projective A-modules of rank r.*

(1) *If* $\dim A = 0$, *then $Q \cong P$. If* $\dim A = 1$, *then $Q \cong P$ if and only if $\wedge^r Q \cong \wedge^r P$.*

(2) *If* $\dim A \le 1$ *and $\nu: \wedge^r Q \xrightarrow{\sim} \wedge^r P$ is an isomorphism, then there exists an isomorphism $\alpha: Q \xrightarrow{\sim} P$ such that $\wedge^r \alpha = \nu$.*

Proof. After reducing to the case where P and Q have constant rank, (1) follows immediately from Theorem 2.4 (Serre). For (2), we reduce to the case of constant rank, and then without loss of generality we may assume that $r \geq 2$. By Theorem 2.4 (Serre), write $Q = Q' \oplus A$. By (1), Let α_0: $Q' \oplus A \xrightarrow{\sim} P$ be any isomorphism, and consider $(\wedge^r \alpha_0)^{-1}\nu \in \mathrm{GL}(\wedge^r Q) = A^\times$, say given by $a \in A^\times$. Then let $\alpha = \alpha_0 \gamma$ where $\gamma \in \mathrm{GL}(Q' \oplus A)$ is $(q, x) \mapsto (q, ax)$, for $q \in Q'$ and $x \in A$. ∎

The following is a typical application of the Eisenbud–Evans theorem.

LEMMA 2.6. *Let A be a Noetherian ring, and Q be a rank r projective module with a unimodular element, say $Q = Q' \oplus A$. Suppose $\tilde{I}$ and I are comaximal ideals such that $r \geq 2 + \dim A/\tilde{I}$, and we have a surjection h: $Q \twoheadrightarrow I$.*

Then there exists $\phi \in \mathrm{SL}(Q)$ such that $h\phi(Q')$ and $\tilde{I}$ are comaximal, and for $\mathfrak{p}$ prime, $\mathfrak{p} \supseteq h\phi(Q')$ implies $\mathfrak{p} \supseteq I$ or $\operatorname{ht} \mathfrak{p} \geq r - 1$. Thus, in particular, if $\dim A/I \leq \delta$ and $r \geq \dim A + 1 - \delta$, then $\dim A/h\phi(Q') \leq \delta$.

Proof. Let X_1 and X_2 be the subsets of Spec A defined by $X_1 = \{\mathfrak{p}: \mathfrak{p} \supseteq \tilde{I}\}$ and $X_2 = \{\mathfrak{p}: \mathfrak{p} \not\supseteq I \text{ and } \operatorname{ht} \mathfrak{p} \leq r(\mathfrak{p}) - 2\}$. Let $X = X_1 \cup X_2$, and define a function on X by

$$d(\mathfrak{p}) = \begin{cases} \dim A/\mathfrak{p} & \text{if } \mathfrak{p} \in X_1,\ \mathfrak{p} \notin X_2 \\ r(\mathfrak{p}) - 2 - \operatorname{ht} \mathfrak{p} & \text{if } \mathfrak{p} \notin X_1,\ \mathfrak{p} \in X_2 \\ \max\{\dim A/\mathfrak{p},\ r(\mathfrak{p}) - 2 - \operatorname{ht} \mathfrak{p}\} & \text{if } \mathfrak{p} \in X_1 \cap X_2. \end{cases}$$

By Examples 2.1, d is a generalized dimension function. Note that for all $\mathfrak{p} \in X$, $\operatorname{rank}_{A_\mathfrak{p}} Q'_\mathfrak{p} = r(\mathfrak{p}) - 1 \geq 1 + d(\mathfrak{p})$. And $h = (h', a) \in Q^* \cong (Q')^* \oplus A$ is basic on Spec $A - V(I)$; in particular, it is basic on X. Applying Theorem 2.2 (Eisenbud–Evans) to (h', a), we get $f' \in (Q')^*$ such that $h' + af'$ is basic on X. Let ϕ be the automorphism of $Q = Q' \oplus A$ given by $(q, b) \mapsto (q, b + f'(q))$, for $q \in Q'$ and $b \in A$. Clearly $\phi \in \mathrm{SL}(Q)$. Also notice $h\phi|_{Q'} = h' + af'$, so that if $\mathfrak{p} \supseteq h\phi(Q')$, then $\mathfrak{p} \notin X$. In particular, $\mathfrak{p} \supseteq h\phi(Q')$ implies $\mathfrak{p} \not\supseteq \tilde{I}$, so that $h\phi(Q')$ and $\tilde{I}$ are comaximal. And since $\mathfrak{p} \supseteq h\phi(Q')$ implies $\mathfrak{p} \notin X_2$, we have either $\mathfrak{p} \supseteq I$ or $\operatorname{ht} \mathfrak{p} \geq r - 1$. The final statement of the lemma follows directly from this. ∎

Remark 2.7. For any integer $k \geq 1$, we could have applied Theorem 2.2 (Eisenbud–Evans) for the submodule $I^k(Q')^* \subseteq (Q')^*$. The resulting ϕ would additionally have the property that it induces the identity on Q/I^kQ.

Many of our addition and subtraction results will use the following theorem, which is based on Mandal's Theorem [Ma, Theorem 2.1]. This extension is due to Mandal and Sridharan [MS, Theorem 2.3].

THEOREM 2.8 (Mandal–Sridharan). *Let A be a Noetherian ring and $R = A[t]$. Suppose I' and I'' are comaximal ideals of R such that I' contains a monic polynomial and $I'' = I''(0)R$ is an extended ideal. Suppose P is a projective A-module of rank $r \geq 2 + \dim R/I'$. If there exists surjections f: $P \twoheadrightarrow I'(0) \cap I''(0)$ and Φ: $P[t]/I'P[t] \twoheadrightarrow I'/(I')^2$ such that $f \bmod I'(0) = \Phi(0)$, then there is a surjection Ψ: $P[t] \to I' \cap I''$ such that $\Psi(0) = f$.*

Remark 2.9. Note that since I' and I'' are comaximal, $I' \cap I''/I'(I' \cap I'') = I'/(I')^2$. So in the statement above, $f \bmod I'(0)$ means the map $f \otimes_A A/I'(0)$: $P/I'(0)P \twoheadrightarrow I'(0)/(I'(0))^2$.

3. ADDITION

As mentioned in Section 1, Mohan Kumar pointed out that certain addition results can be proved directly from the Eisenbud–Evans Theorem (oral communication). In Theorem 3.2, we will show that this technique gives us rather tight control on the surjection thus found. We will then use Theorem 2.8 (Mandal–Sridharan) to prove other addition results, Theorem 3.5 and Theorem 3.6. We begin by recalling the following well known result of Mohan Kumar.

PROPOSITION 3.1 [MK1, Lemma 1]. *Let A be a ring, $I, I', J \subseteq A$ ideals such that I is finitely generated, and $J \subseteq I$. Suppose $I = I' + JI$. Then for any $z \in A$, there exists $z' \in z + J$ such that $I + zA = I' + z'A$.*

Proof. Let "bar" denote mod I'. Since $\bar{I} = \bar{J}\bar{I}$, by Nakayama's Lemma (e.g., [Mat]), there exists $h \in J$ such that $(1 - \bar{h})\bar{I} = 0$. So $I = I' + hI = I' + hA$ and $(1 - h)h \in I'$. Let $z' = z + h - zh$. Clearly $I' + z'A \subseteq I + zA$. For the opposite inclusion, it suffices to show $h, z \in I' + z'A$. But using $1 = (1 - z)(1 - h) + z'$, we get $h = (1 - z)(1 - h)h + z'h \in I' + z'A$; using this, $z = z' - h + zh \in I' + z'A$ as well, and we are done. ∎

THEOREM 3.2. *Let A be a Noetherian ring of dimension n. Let $I \subseteq J \subseteq A$ be ideals, and let Q_0 and Q_1 be projective A-modules of rank r, each having a unimodular element. Suppose we have one of the following sets of conditions.*

(1) $\dim A/I = 0$ *and* $r \geq n + 1$.

(2) $\dim A/I \leq 1$, $r \geq n$, *and there exists an isomorphism* η: $\wedge^r Q_0 \xrightarrow{\sim} \wedge^r Q_1$.

If there exists a surjection h_1: $Q_1 \twoheadrightarrow J$, then there exists a surjection h_0: $Q_0 \twoheadrightarrow J$ and an isomorphism α: $Q_0/IQ_0 \xrightarrow{\sim} Q_1/IQ_1$ such that the following *diagram commutes.*

$$\begin{tikzcd} Q_0/IQ_0 \arrow[rr, "\alpha"] \arrow[dr, "h_0 \otimes_A A/I"'] & & Q_1/IQ_1 \arrow[dl, "h_1 \otimes_A A/I"] \\ & J/IJ & \end{tikzcd}$$

In case (2), *we also will have* $\wedge^r \alpha = \eta \otimes_A A/I$.

COROLLARY 3.3. *Let A be a Noetherian ring of dimension n, $I \subseteq A$ an ideal, and Q_0 and Q_1 projective A-modules of rank r, each having a unimodular element. Suppose we have one of the following sets of conditions.*

(1) $\dim A/I = 0$ *and* $r \geq n + 1$.

(2) $\dim A/I \leq 1$, $r \geq n$, *and* $\wedge^r Q_0 \cong \wedge^r Q_1$.

If there exists a surjection $Q_1 \twoheadrightarrow I$, then there exists a surjection $Q_0 \twoheadrightarrow I$.

As mentioned in Section 1, Sridharan proved this result for $Q_1 = A^n$ and $r = \operatorname{ht} I = n \geq 3$ [RS1, Theorem 1].

Remarks 3.4. (1) If we have $I \subseteq A$, $Q_1 \twoheadrightarrow A$, and $Q_1 \twoheadrightarrow I$, as in Corollary 3.3, then we can view Corollary 3.3 as an addition result, taking $Q_0 \twoheadrightarrow A$ and producing $Q_0 \twoheadrightarrow I$.

(2) Aside from the case $r = n = 2$, Corollary 3.3 is a special case of Theorem 3.5. In Theorem 3.2, however, we get some control over the surjection we find, unlike in Theorem 3.5.

Proof of Theorem 3.2. By first reducing to the case of constant rank, we may assume without loss of generality that $r \geq 2$. Let δ be the maximum allowable value of $\dim A/I$, i.e., 0 if the first set of conditions holds, and 1 if the second set holds. Write $Q_i = Q_i' \oplus Ax_i$ for $i = 0, 1$, with each Q_i' a rank $r - 1$ projective module. Applying Lemma 2.6 (with $\tilde{I} = A$), there exists $\phi \in \mathrm{SL}(Q_1)$ such that if $K = h_1\phi(Q_1')$, $\dim A/K \leq \delta$. Since $\dim A/IK \leq \delta$, apply Corollary 2.5(1) to get an isomorphism $\zeta'\colon Q_0'/IKQ_0' \xrightarrow{\sim} Q_1'/IKQ_1'$. When $\delta = 1$, by Corollary 2.5(2) we can additionally assume that the following diagram commutes (where the vertical maps are the isomorphisms induced by $Q_i = Q_i' \oplus Ax_i \cong Q_i' \oplus A$).

$$\begin{tikzcd} \wedge^{r-1} Q_0'/IKQ_0' \arrow[r, "\wedge^{r-1}\zeta'"] \arrow[d, "\cong"'] & \wedge^{r-1} Q_1'/IKQ_1' \arrow[d, "\cong"] \\ \wedge^r Q_0/IKQ_0 \arrow[r, "\eta \otimes_A A/IK"] & \wedge^r Q_1/IKQ_1 \end{tikzcd}$$

Define an isomorphism $\zeta\colon Q_0/IQ_0 \xrightarrow{\sim} Q_1/IQ_1$ as in the following diagram.

$$\begin{tikzcd}
Q_0/IQ_0 \arrow[r, equal] \arrow[d, "\zeta"'] & Q_0'/IQ_0' \arrow[d, "\zeta' \otimes_{A/IK} A/I"'] \oplus (A/I)x_0 \arrow[d, "x_0 \mapsto x_1"] \\
Q_1/IQ_1 \arrow[r, equal] & Q_1'/IQ_1' \oplus (A/I)x_1
\end{tikzcd}$$

Let $\alpha = (\phi \otimes_A A/I)\zeta$. When $\delta = 1$, $\wedge^r\alpha = \wedge^r\zeta = \wedge^{r-1}(\zeta' \otimes_{A/IK} A/I) = \eta \otimes_A A/I$. Now, since $K = h_1\phi(Q_1')$, we get a surjection $[(h_1\phi)|_{Q_1'} \otimes_A A/IK]\zeta'\colon Q_0'/IKQ_0' \twoheadrightarrow K/IK^2$. Lift this map to some $h_0'\colon Q_0' \to K$, and apply Proposition 3.1 with $I = K$, $I' = \operatorname{im} h_0'$, $J = IK$, and $z = h_1\phi(x_1)$. We get $z' \in h_1\phi(x_1) + IK$ such that $\operatorname{im} h_0' + Az' = K + A(h_1\phi(x_1)) = J$. Define $h_0\colon Q_0 = Q_0' \oplus Ax_0 \twoheadrightarrow J$ which extends $h_0'\colon Q_0' \to K \subseteq J$ by sending $x_0 \mapsto z'$. Since $z' \in h_1\phi(x_1) + IK \subseteq h_1\phi(x_1) + IJ$, we have the following diagram commutative, as desired.

$$\begin{tikzcd}
Q_0/IQ_0 \arrow[r, "\zeta"'] \arrow[rr, bend left, "\alpha"] \arrow[dr, "h_0 \otimes_A A/I"'] & Q_1/IQ_1 \arrow[r, "\phi \otimes_A A/I"'] & Q_1/IQ_1 \arrow[dl, "h_1 \otimes_A A/I"] \\
& J/IJ &
\end{tikzcd}$$

▮

Now we will use Theorem 2.8 (Mandal–Sridharan) to prove more addition results. For a partial converse, see Theorem 4.1 or Theorem 6.1.

THEOREM 3.5. *Let A be a Noetherian ring of dimension n. Let P and Q be rank r projective A-modules, Q having a unimodular element. Suppose $\tilde{I}$ and I are comaximal ideals of A with* $\dim A/\tilde{I} \le r-2$. *Suppose we have one of the following sets of conditions.*

(1) $\dim A/I = 0$ *and* $r \ge n+1$.

(2) $\dim A/I \le 1$, $r \ge \max\{n, 3\}$, *and* $\wedge^r P \cong \wedge^r Q$.

If there exist surjections $f\colon P \twoheadrightarrow \tilde{I}$ and $h\colon Q \twoheadrightarrow I$, then there exists a surjection $P \twoheadrightarrow \tilde{I} \cap I$.

Proof. As in the proof of Theorem 3.2, let δ be the maximum allowable value of $\dim A/I$, i.e., 0 if the first set of conditions holds, and 1 if the second set holds. Note that when $r \ge n+1$, the case of $r = 1$ is trivial because $P \cong Q \cong A$ by Corollary 2.5(1), and so $\tilde{I}$ and I principal imply $\tilde{I} \cap I = \tilde{I}I$ is principal. So when $\delta = 0$, reducing to the case of constant rank, we may assume $r \ge \max\{n+1, 2\}$. We will use that $r \ge 2 + \delta$ and

$r \geq n + 1 - \delta$. Write $Q = Q' \oplus Ax$. Without loss of generality, we may assume that $h(Q')$ and $\tilde{I}$ are comaximal, and that $\dim A/h(Q') \leq \delta$. Indeed, we apply Lemma 2.6 to get an automorphism ϕ of Q such that $h\phi(Q')$ has the desired properties, and we replace h by $h\phi$.

Let $I' \subseteq A[t]$ be the ideal $h(Q')A[t] + (t+1)A[t]$. Consider the surjection $Q[t] = Q'[t] \oplus A[t]x \twoheadrightarrow I'$ mapping $Q'[t]$ by $h|_{Q'} \otimes_A A[t]$ and mapping $x \mapsto t+1$. This induces a surjection Φ: $P[t]/I'P[t] \cong Q[t]/I'Q[t] \twoheadrightarrow I'/(I')^2$, where we get the first isomorphism from Corollary 2.5(1) since $\dim A[t]/I' = \dim A/h(Q') \leq \delta$.

Let $I'' \subseteq A[t]$ be the extended ideal $\tilde{I}A[t]$. Since $h(Q')$ and $\tilde{I}$ are comaximal, so are I' and I''. Using $r \geq 2 + \delta \geq 2 + \dim A[t]/I'$, we apply Theorem 2.8 (Mandal–Sridharan) to f: $P \twoheadrightarrow \tilde{I} = I'(0) \cap I''(0)$ and the above surjection Φ, to get a surjection $P[t] \twoheadrightarrow I' \cap I''$. (The condition $f \bmod I'(0) = \Phi(0)$ is trivially satisfied since $I'(0) = A$.) We specialize this surjection at $t = h(x) - 1$ to get our desired surjection $P \twoheadrightarrow I'(h(x)-1) \cap I''(h(x)-1) = I \cap \tilde{I}$. ∎

THEOREM 3.6. *Let A be a Noetherian ring of dimension n. Let Q be a rank r projective A-module having a unimodular element. Suppose $\tilde{I}$ and I are comaximal ideals of A with $\dim A/\tilde{I} \leq r-2$ and $\dim A/I \leq r-2$. If $2r \geq n+3$ and there exist surjections $Q \twoheadrightarrow \tilde{I}$ and $Q \twoheadrightarrow I$, then there exists a surjection $Q \twoheadrightarrow \tilde{I} \cap I$.*

Theorem 3.6 was proved by Sridharan [RS1, Theorem 4] for $Q = A^r$, $\operatorname{ht} I = \operatorname{ht} \tilde{I} = r$, $\dim A/\tilde{I} = \dim A/I = n - r$, and $2r \geq n+3$. Note that these conditions in fact imply $\dim A/\tilde{I} \leq r-3$ and $\dim A/I \leq r-3$. Even earlier, this theorem was proved by Mohan Kumar [MK2, Corollary 1] for A a reduced affine ring of dimension n over a field which is finite or algebraically closed, $Q = A^n$, and $\tilde{I}$ and I complete intersection ideals.

Proof. Write $Q = Q' \oplus Ax$. As in the proof of Theorem 3.5, by applying Lemma 2.6, we can find a surjection g_1: $Q \twoheadrightarrow \tilde{I}$ such that $g_1(Q')$ and I are comaximal and $\dim A/g_1(Q') \leq r-2$. Applying Lemma 2.6 again, we can find a surjection g_2: $Q \twoheadrightarrow I$ such that $g_1(Q')$ and $g_2(Q')$ are comaximal and $\dim A/g_2(Q') \leq r-2$.

For $i = 1, 2$, let F_i be the polynomial $t^2 + (g_i(x) - 2)t + 1 \in A[t]$, and let $J_i \subseteq A[t]$ be the ideal $g_i(Q')A[t] + F_iA[t]$. Since F_i is monic, $\dim A[t]/J_i = \dim A/g_i(Q') \leq r-2$. We have surjections $Q[t] \twoheadrightarrow J_i$ induced by $g_i|_{Q'} \otimes_A A[t]$ and $x \mapsto F_i$. Let $I' = J_1 \cap J_2$, which contains the monic polynomial F_1F_2. Since $g_1(Q')$ and $g_2(Q')$ are comaximal, so are J_1 and J_2. By the Chinese Remainder Theorem, the above surjections induce $Q[t]/I'Q[t] \cong Q[t]/J_1Q[t] \oplus Q[t]/J_2Q[t] \twoheadrightarrow J_1/J_1^2 \oplus J_2/J_2^2 \cong I'/(I')^2$. We can now apply Theorem 2.8 (Mandal–Sridharan) (with $I'' = A[t]$) to get a surjection $Q[t] \twoheadrightarrow I'$. Specializing this map at $t = 1$, we get our desired surjection $Q \twoheadrightarrow J_1(1) \cap J_2(1) = \tilde{I} \cap I$. ∎

4. THE SUBTRACTION PRINCIPLE

In a Noetherian ring A with P a projective module, subtraction is always possible when $\operatorname{rank} P \geq 1 + \dim A$, as we see in the following theorem. Once we have dealt with this case, the purpose of this section is to establish a principle for subtraction when $\operatorname{rank} P = \dim A$, which we will do in Theorem 4.5.

THEOREM 4.1. *Let A be a Noetherian ring, P a projective A-module with* $\operatorname{rank} P \geq 1 + \dim A$. *Suppose $\tilde{I}$ and I are comaximal ideals of A. If there exists a surjection $P \twoheadrightarrow \tilde{I} \cap I$, then there exists a surjection $P \twoheadrightarrow \tilde{I}$.*

Proof. Let $g\colon P \twoheadrightarrow \tilde{I} \cap I$ be a surjective map; we think of g as an element of P^*. Let $X = \operatorname{Spec} A - V(\tilde{I})$. Applying Theorem 2.2 (Eisenbud–Evans) to $(g, 1)$ and $\tilde{I}^2 P^* \subseteq P^*$, we get $h \in \tilde{I}^2 P^*$ such that $f = g + h$ is basic on X. We claim $f\colon P \to A$ has image $\tilde{I}$. Since g and h map into $\tilde{I}$, $\operatorname{im} f \subseteq \tilde{I}$. Now, since $h \in \tilde{I}^2 P^*$, $\bar{f} = \bar{g}\colon P/\tilde{I}P \twoheadrightarrow \tilde{I}/\tilde{I}^2$, which is surjective. So if $\mathfrak{p} \supseteq \tilde{I}$, $\bar{f}_{\mathfrak{p}}\colon P_{\mathfrak{p}}/\tilde{I}P_{\mathfrak{p}} \twoheadrightarrow \tilde{I}_{\mathfrak{p}}/\tilde{I}_{\mathfrak{p}}^2$ is surjective, and by Nakayama's Lemma, so is $f_{\mathfrak{p}}\colon P_{\mathfrak{p}} \twoheadrightarrow \tilde{I}_{\mathfrak{p}}$. But if $\mathfrak{p} \not\supseteq \tilde{I}$, then since f is basic at $\mathfrak{p}$, $f_{\mathfrak{p}}\colon P_{\mathfrak{p}} \twoheadrightarrow A_{\mathfrak{p}} = \tilde{I}_{\mathfrak{p}}$. So $f\colon P \twoheadrightarrow \tilde{I}$ is surjective at all primes, and we are done. ▮

Remark 4.2. In fact, Theorem 4.1 holds under the slightly weaker hypothesis that $\operatorname{rank} P \geq 1 + \text{j-dim}(\operatorname{Spec} A - V(\tilde{I}))$. See [Sw] for the definition of j-dim and details on applying the Eisenbud–Evans Theorem with generalized dimension functions which involve j-dim.

PROPOSITION 4.3. *Let B be a Noetherian ring with* $\dim B \leq 1$, *Q a projective B-module, and $B \twoheadrightarrow \overline{B}$ a surjective ring homomorphism. Then the natural map* $\mathrm{SL}(Q) \to \mathrm{SL}(Q \otimes_B \overline{B})$ *is surjective.*

Remark 4.4. The hypotheses that B is Noetherian and $\dim B \leq 1$ can be weakened to $\operatorname{psr} B \leq 2$. For details on projective stable range, see [Sw].

Proof. Without loss of generality, we may assume Q has constant rank r and $Q \neq 0$. Then by Theorem 2.4 (Serre), $Q \cong Q_1 \oplus Q_2$ with Q_1 of rank 1 and Q_2 free of rank $r - 1$. We induct on r, $r = 1$ being trivial. We write $\overline{Q}$ for $Q \otimes_B \overline{B}$ and $\overline{Q}_i$ for $Q_i \otimes_B \overline{B}$. Let I be the kernel of $B \twoheadrightarrow \overline{B}$.

For $r = 2$, write an arbitrary element of $\mathrm{SL}(\overline{Q})$ as a 2×2 matrix $(\bar{b}_{ij})$, with $\bar{b}_{ij} \in \operatorname{Hom}_{\overline{B}}(\overline{Q}_j, \overline{Q}_i)$. Then $1 = \det(\bar{b}_{ij}) = \bar{b}_{11}\bar{b}_{22} - \bar{b}_{12}\bar{b}_{21}$, where $\bar{b}_{12}\bar{b}_{21}$ indicates the composition in $\operatorname{Hom}_{\overline{B}}(\overline{Q}_1, \overline{Q}_1) = \overline{B}$. Choose any lift of each $\bar{b}_{ij}$ to get $(b_{ij}) \in \operatorname{End}_B(Q)$. Let $x = 1 - b_{11}b_{22} + b_{12}b_{21} \in I$; so $(b_{21} \;\; b_{22} \;\; x)$ is a unimodular row. By Theorem 2.2 (Eisenbud–Evans), there exist $b'_{21} \in \operatorname{Hom}_B(Q_1, Q_2)$ and $b'_{22} \in \operatorname{Hom}_B(Q_2, Q_2)$ such that $(b_{21} + xb'_{21} \;\; b_{22} + xb'_{22})$ is unimodular. Replacing b_{21} and b_{22} with $b_{21} +$

xb'_{21} and $b_{22} + xb'_{22}$, we may assume $(b_{21}\ \ b_{22})$ is unimodular, say $c_{11}b_{22} + c_{12}b_{21} = 1$ for $c_{11} \in B$ and $c_{12} \in \operatorname{Hom}_B(Q_2, Q_1)$. Consider the matrix

$$\begin{pmatrix} b_{11} + xc_{11} & b_{12} - xc_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Since $x = 1 - b_{11}b_{22} + b_{12}b_{21} \in I$, this matrix also lifts $(\bar{b}_{ij})$, and its determinant is $b_{11}b_{22} - b_{12}b_{21} + x(c_{11}b_{22} + c_{12}b_{21}) = 1$, which completes the $r = 2$ case.

We now inductively prove the case $r \geq 3$. As above, let $(\bar{b}_{ij})$ be an arbitrary element of $\mathrm{SL}(\bar{Q})$, $\bar{b}_{ij} \in \operatorname{Hom}_{\bar{B}}(\bar{Q}_j, \bar{Q}_i)$. Now, $(\bar{b}_{11}\ \ \bar{b}_{12}) \in \bar{B} \oplus \operatorname{Hom}_{\bar{B}}(\bar{Q}_2, \bar{Q}_1)$ is a unimodular row. (Indeed, locally it is the bottom row of a matrix in $\mathrm{SL}(\bar{B}^r_{\mathfrak{p}})$.) Since $r - 1 \geq 1 + \dim \bar{B}$ we can apply Theorem 2.2 (Eisenbud–Evans) to find $\bar{b}'_{12} \in \operatorname{Hom}_{\bar{B}}(\bar{Q}_2, \bar{Q}_1)$ such that $\bar{b}_{12} + \bar{b}_{11}\bar{b}'_{12}$ is unimodular. Consider the equation

$$\begin{pmatrix} \bar{b}_{11} & \bar{b}_{12} \\ \bar{b}_{21} & \bar{b}_{22} \end{pmatrix} = \begin{pmatrix} \bar{b}_{11} & \bar{b}_{12} + \bar{b}_{11}\bar{b}'_{12} \\ \bar{b}_{21} & \bar{b}_{22} + \bar{b}_{21}\bar{b}'_{12} \end{pmatrix} \begin{pmatrix} 1 & -\bar{b}'_{12} \\ 0 & 1 \end{pmatrix}.$$

The last matrix is elementary, and thus lifts to an element of $\mathrm{SL}(Q)$. So replacing $(\bar{b}_{ij})$, we may assume that $\bar{b}_{12}$ is unimodular, say $\bar{c}_{21} \in \operatorname{Hom}_{\bar{B}}(\bar{Q}_1, \bar{Q}_2)$ with $\bar{b}_{12}\bar{c}_{21} = 1$. Now consider the equation

$$\begin{pmatrix} \bar{b}_{11} & \bar{b}_{12} \\ \bar{b}_{21} & \bar{b}_{22} \end{pmatrix} = \begin{pmatrix} 1 & \bar{b}_{12} \\ \bar{b}_{21} - \bar{b}_{22}\bar{c}_{21}\big(\bar{b}_{11} - 1\big) & \bar{b}_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{c}_{21}\big(\bar{b}_{11} - 1\big) & 1 \end{pmatrix}.$$

As above, the second matrix can be lifted to $\mathrm{SL}(Q)$, so again replacing $(\bar{b}_{ij})$, we may assume that $\bar{b}_{11} = 1$. Finally, we have the equation

$$\begin{pmatrix} 1 & \bar{b}_{12} \\ \bar{b}_{21} & \bar{b}_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \bar{b}_{21} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \bar{b}_{22} - \bar{b}_{21}\bar{b}_{12} \end{pmatrix} \begin{pmatrix} 1 & \bar{b}_{12} \\ 0 & 1 \end{pmatrix}.$$

Again, the outer matrices can be lifted to $\mathrm{SL}(Q)$, and by our induction hypothesis, the middle matrix can also be lifted to $\mathrm{SL}(Q)$, so we are done. ∎

Now we are ready to state and prove our basic principle for subtraction.

THEOREM 4.5 (Subtraction Principle). *Let A be a Noetherian ring of dimension $n \geq 3$. Let P and Q be projective modules of rank n, Q having a unimodular element, with $\wedge^n Q \cong \wedge^n P$. Let $\tilde{I}$ and I be comaximal ideals with* $\dim A/\tilde{I} \leq n - 2$ *and* $\dim A/I \leq 1$. *Suppose there exist surjections*

$h\colon Q \twoheadrightarrow I$ and $g\colon P \twoheadrightarrow \tilde{I} \cap I$ and an isomorphism $\gamma\colon Q/IQ \xrightarrow{\sim} P/IP$ such that $\wedge^n \gamma = \nu \otimes_A A/I$ for some isomorphism $\nu\colon \wedge^n Q \xrightarrow{\sim} \wedge^n P$, and such that the following diagram commutes.

$$\begin{tikzcd} Q/IQ \arrow[r, "\gamma"] \arrow[d, "h \otimes_A A/I"'] & P/IP \arrow[d, "g \otimes_A A/I"] \\ I/I^2 \arrow[r, equal] & \tilde{I} \cap I/I(\tilde{I} \cap I) \end{tikzcd}$$

Then there exists a surjection $P \twoheadrightarrow \tilde{I}$.

Proof. Write $Q = Q' \oplus Ax$. By Lemma 2.6, we get $\phi \in \mathrm{SL}(Q)$ such that $h\phi(Q')$ and $\tilde{I}$ are comaximal, and $\dim A/h\phi(Q') \le 1$. Replacing h by $h\phi$ and γ by $\gamma(\phi \otimes_A A/I)$, we may assume $h(Q')$ and $\tilde{I}$ are comaximal and $\dim A/h(Q') \le 1$. Here we use that ϕ has determinant 1, so that $\wedge^n \gamma = \nu \otimes_A A/I$ is preserved.

Let $I', I'' \subseteq A[t]$ be the ideals $I' = h(Q')A[t] + (t + h(x))A[t]$ and $I'' = \tilde{I}A[t]$. Since $h(Q')$ and $\tilde{I}$ are comaximal, so are I' and I''. Note that $I'(0) = I$ and $A[t]/I' \cong A/h(Q')$ has dimension ≤ 1.

By Corollary 2.5(2), choose $\beta\colon Q[t]/I'Q[t] \xrightarrow{\sim} P[t]/I'P[t]$ to be an isomorphism such that $\wedge^n \beta = \nu \otimes_A A[t]/I'$. Since $\wedge^n \beta(0) = \nu \otimes_A A/I = \wedge^n \gamma$, we have $\gamma^{-1}\beta(0) \in \mathrm{SL}(Q/IQ)$. Applying Proposition 4.3 to the surjection $A[t]/I' \twoheadrightarrow A/I$ induced by $t \mapsto 0$, we get that $\mathrm{SL}(Q[t]/I'Q[t]) \twoheadrightarrow \mathrm{SL}(Q/IQ)$ is surjective. So let $\Lambda \in \mathrm{SL}(Q[t]/I'Q[t])$ be such that $\Lambda(0) = \gamma^{-1}\beta(0)$.

Let $H\colon Q[t]/I'Q[t] \twoheadrightarrow I'/(I')^2$ be the map induced by mapping $Q'[t]$ by $h \otimes_A A[t]$ and sending $x \mapsto t + h(x)$. Then $H(0) = h \otimes_A A/I$. Consider the surjections $g\colon P \twoheadrightarrow \tilde{I} \cap I$ and $H\Lambda\beta^{-1}\colon P[t]/I'P[t] \twoheadrightarrow I'/(I')^2$. We then have $(H\Lambda\beta^{-1})(0) = (h \otimes_A A/I)\gamma^{-1}\beta(0)\beta^{-1}(0) = (h \otimes_A A/I)\gamma^{-1} = g \otimes_A A/I$. So we can apply Theorem 2.8 (Mandal–Sridharan) to get a surjective map $P[t] \twoheadrightarrow I' \cap I''$ which we specialize at $t = 1 - h(x)$ to get the desired surjection $P \twoheadrightarrow A \cap \tilde{I} = \tilde{I}$. ∎

5. PREPARATION FOR SUBTRACTION

This section contains the tools we need to prove our main subtraction result, Theorem 6.1, from our subtraction principle, Theorem 4.5. In the notation of Theorem 4.5, recall that we need to have surjections $h\colon Q \twoheadrightarrow I$ and $g\colon P \twoheadrightarrow \tilde{I} \cap I$ and an isomorphism $\gamma\colon Q/IQ \xrightarrow{\sim} P/IP$ such that $(g \otimes_A A/I)\gamma = h \otimes_A A/I$, and $\wedge^n \gamma = \nu \otimes_A A/I$ for some isomorphism $\nu\colon \wedge^n Q \xrightarrow{\sim} \wedge^n P$. Lemma 5.1 and Proposition 5.4 will allow us to find γ such that $(g \otimes_A A/I)\gamma = h \otimes_A A/I$. We will use Proposition 5.6 then to alter h and γ so that we will also have $\wedge^n\gamma$ with the desired form. Finally, Proposition 5.7 will allow us to push our subtraction results a bit farther. (As we will see in Remark 6.3, even without Proposition 5.7, we will get all the applications of Section 1.)

LEMMA 5.1. *Let A be a Noetherian ring,* $\dim A \leq 1$, *P and M projective A-modules. If $f, g\colon P \twoheadrightarrow M$ are surjections, then there exists $\theta \in \mathrm{GL}(P)$ such that $f\theta = g$. Further, if* $\operatorname{rank} P \geq 1 + \operatorname{rank} M$, *we can choose $\theta \in \mathrm{SL}(P)$.*

Proof. Let $K_1 = \ker f$ and $K_2 = \ker g$, and consider the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & K_2 \arrow[r, hook] \arrow[d, dashed, "\alpha"', "\cong"] & P \arrow[r, "g"'] \arrow[d, dashed, "\theta"] & M \arrow[r] \arrow[l, dotted, bend right, "t"'] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_1 \arrow[r, hook] & P \arrow[r, "f"'] & M \arrow[r] \arrow[l, dotted, bend right, "s"'] & 0 \end{tikzcd}$$

Since M is projective, we choose any splitting maps s and t as indicated. Now, $P \cong M \oplus K_1 \cong M \oplus K_2$. So by Corollary 2.5(1), we can find an isomorphism $\alpha\colon K_1 \xrightarrow{\sim} K_2$. This allows us to find θ which makes the diagram commute. Specifically, we select $\theta = sg + \alpha(1 - tg)$. By the snake lemma, $\theta \in \mathrm{GL}(P)$, and clearly $f\theta = g$.

Now suppose $\operatorname{rank} P \geq 1 + \operatorname{rank} M$. Since $\operatorname{rank} K_1 \geq 1$, we have that $\mathrm{GL}(\wedge^{\operatorname{rank} K_1} K_1) = A^\times = \mathrm{GL}(\wedge^{\operatorname{rank} P} P)$. Applying Corollary 2.5(2), find $\beta \in \mathrm{GL}(K_1)$ with $\det \beta = \det \theta^{-1}$. Let $\psi = \beta \oplus 1_M \in \mathrm{GL}(P)$, i.e., $\psi = sf + \beta(1 - sf)$. Then $\det \psi = \det \beta = \det \theta^{-1}$ and $f\psi = f$. Letting $\theta' = \psi\theta \in \mathrm{SL}(P)$, we have $f\theta' = f\theta = g$, as desired. ∎

We recall the following consequence of Nakayama's Lemma. (See, e.g., [L, I.1.6].)

PROPOSITION 5.2. *Let A be a ring, $I \subseteq$* $\operatorname{rad} A$ *an ideal, P and Q projective modules. Let "bar" denote* mod I.

(1) *Any isomorphism $\overline{P} \xrightarrow{\sim} \overline{Q}$ can be lifted to an isomorphism $P \xrightarrow{\sim} Q$.*

(2) *For $\theta\colon P \to Q$ a homomorphism, if $\overline{\theta}$ is an isomorphism, then so is θ.*

LEMMA 5.3. *Let A be any ring, $I \subseteq$* $\operatorname{rad} A$ *an ideal, M an A-module, P a projective A-module, $f, g\colon P \to M$ maps with f surjective. Let "bar" denote* mod I. *Suppose there exists* $\bar{\theta} \in \mathrm{GL}(\bar{P})$ *such that* $\bar{f}\bar{\theta} = \bar{g}$. *Then there exists* $\theta \in \mathrm{GL}(P)$ *which lifts* $\bar{\theta}$, *and such that* $f\theta = g$.

Proof. By Proposition 5.2(1), let $\theta_0 \in \mathrm{GL}(P)$ lift $\bar{\theta}$. Then $\overline{f\theta_0 - g} = 0$, so $f\theta_0 - g$ has image in IM. Since P is projective, we can find $\alpha\colon P \to IP$ making the following diagram commute.

$$\begin{tikzcd} & IP \arrow[d, "f"] \\ P \arrow[ur, "\alpha"] \arrow[r, "f\theta_0 - g"'] & IM \end{tikzcd}$$

Let $\theta = \theta_0 - \alpha$, which also lifts $\bar{\theta}$. So $f\theta = f\theta_0 - (f\theta_0 - g) = g$, and by Proposition 5.2(2), $\theta \in \mathrm{GL}(P)$. ∎

PROPOSITION 5.4. *If A is an Artinian ring, M a module, P a projective module, and $f, g\colon P \twoheadrightarrow M$ surjections, then there exists $\theta \in \mathrm{GL}(P)$ such that $f\theta = g$.*

Proof. Let "bar" denote mod rad A. Since $\bar{A}$ is the product of fields, $\bar{P}$ and $\bar{M}$ are both $\bar{A}$-projective. Apply Lemma 5.1 to get $\bar{\theta} \in \mathrm{GL}(\bar{P})$ such that $\bar{f}\bar{\theta} = \bar{g}$. Then apply Lemma 5.3 to get the desired θ. ∎

In the next lemma, we will use (1) to prove (2), which will be used to prove Proposition 5.6.

LEMMA 5.5. *Let A be a ring, M an $n \times n$ matrix with entries in A, and $d = \det M$.*

(1) *Let M' be the adjoint of M. Then $\det M' = d^{n-1}$. If $n \geq 2$, then $d^{n-2}M$ is the adjoint of M'.*

(2) *Suppose $n \geq 2$. Write in block form*

$$M = \begin{pmatrix} x & H \\ K & L \end{pmatrix},$$

where $x \in A$, and H, K, and L are $1 \times (n-1), (n-1) \times 1$, and $(n-1) \times (n-1)$ matrices, respectively. Let B be the adjoint of L. Let I [resp. I_0] be the image of the map given by H [resp. HB]: $A^{n-1} \to A$. Then $x^{n-2}\det B \equiv d^{n-2} \bmod I_0$ and $d^{n-2}I \subseteq I_0 \subseteq I$. In particular, $I_0 = I$ if M is invertible.

Proof. Other than the final sentence (which clearly follows from the previous statement), we may prove the lemma for $A = \mathbb{Z}[T_{11}, \ldots, T_{nn}]$ (with T_{ij} indeterminates, $1 \leq i, j \leq n$) and $M = (T_{ij})$. Then we specialize to any matrix over any ring. So without loss of generality, we assume A is a domain, $d \neq 0$, and in the notation of (2), $\det L \neq 0$.

For (1), since $d \cdot \det M' = \det(MM') = \det(d1_n) = d^n$, by our assumptions above, $\det M' = d^{n-1}$. Further, taking $d1_n = MM'$ and right multiplying by the adjoint of M', we get $d(M')^{\mathrm{adj}} = M \cdot \det M' = d^{n-1}M$, so by canceling again, $(M')^{\mathrm{adj}} = d^{n-2}M$, as desired.

For (2), write the adjoint of M in similar block form:

$$M' = \begin{pmatrix} x' & H' \\ K' & L' \end{pmatrix}.$$

We claim that $HB = -H'$. Indeed, we have the equation

$$\begin{pmatrix} d & 0 \\ 0 & d1_{n-1} \end{pmatrix} = d1_n = M'M = \begin{pmatrix} x' & H' \\ K' & L' \end{pmatrix}\begin{pmatrix} x & H \\ K & L \end{pmatrix}.$$

Thus $x'H + H'L = 0$, or $x'H = -H'L$. Right multiplying by B, $x'HB = -\det L \cdot H'$. Note that by standard techniques to compute adjoint matrices, $x' = \det L$; and so by our choice of A and M at the beginning of the proof of this lemma, we can cancel to get $HB = -H'$, as desired. Note that trivially $\mathrm{im}(HB) \subseteq \mathrm{im}(H)$, so we conclude $\mathrm{im}(H') = \mathrm{im}(HB) \subseteq \mathrm{im}(H)$. Repeating this argument for M' and its adjoint $d^{n-2}M$, we conclude that $d^{n-2}\mathrm{im}(H) = \mathrm{im}(d^{n-2}H) \subseteq \mathrm{im}(H')$. Since $I = \mathrm{im}(H)$ and $I_0 = \mathrm{im}(HB) = \mathrm{im}(H')$, we have the desired $d^{n-2}I \subseteq I_0 \subseteq I$.

Finally, the matrix equation above gives us $x'x + H'K = d$. Since $\mathrm{im}(H') = \mathrm{im}(HB) = I_0$, we get that $x'x \equiv d \bmod I_0$. From (1), $\det B = (\det L)^{n-2}$; and we saw above that $x' = \det L$. Thus $x^{n-2}\det B = (xx')^{n-2} \equiv d^{n-2} \bmod I_0$. ∎

We now use Lemma 5.5(2) in the following extension of a result of Sridharan [RS1, Proposition to Theorem 5; RS2, Theorem 1.11].

PROPOSITION 5.6 (Sridharan). *Let A be a ring and Q a projective A-module which has a free direct summand of rank 2. Suppose h: $Q \twoheadrightarrow I$ is a surjection onto an ideal I, and $d \in A/I$ is a unit. Suppose either*

(a) *A is a $\mathbb{Z}$-algebra of finite type*, $\operatorname{rank} Q \geq \dim A$, *and* $\dim A/I \leq 2$,

or

(b) *$d = d_0^2$ for some $d_0 \in A/I$.*

Then there exists $B \in \mathrm{End}(Q)$ such that hB: $Q \twoheadrightarrow I$ is surjective and $\det B \in A$ has image $d^{-1} \in A/I$.

Proof. Consider the case $Q = A^2$. Select a basis of Q, and write $h = (x_1, x_2)$ with respect to this basis. Note that $I = Ax_1 + Ax_2$. Select $x_0 \in A$ which is a preimage of $d \in A/I$. This makes $(x_0 \ \ x_1 \ \ x_2)$ a unimodular row. In case (b), we can choose x_0 to be a square, and by a theorem of

Swan–Towber [ST], there exists a 3×3 matrix M with entries in A such that M has determinant 1 and first row $(x_0 \ \ x_1 \ \ x_2)$. In case (a), $\dim A \leq \operatorname{rank} Q = 2$, so we may choose such an M by [VS, Corollary 18.1, Definition in Sects. 2, 7]. In both cases, let $B \in \operatorname{End}(Q)$ be given by the adjoint of the 2×2 matrix in the lower right corner of M. By Lemma 5.5(2), hB has image I and $x_0 \det B \equiv 1 \bmod I$, so $\det B$ has image d^{-1}, as desired.

For the general case, write $Q = Q_1 \oplus Q_2$ with Q_1 free of rank 2. (In case (a), we apply Lemma 2.6 twice to get $\phi \in \mathrm{SL}(Q)$ with $\dim A/h\phi(Q_2) \leq 2$ so that by replacing h by $h\phi$, we may assume without loss of generality that $\dim A/h(Q_2) \leq 2$.) Let "bar" be reduction mod $h(Q_2)$. We can apply the above case to $\bar{h}$: $\overline{Q}_1 \twoheadrightarrow \bar{I}$ and $d \in A/I = \overline{A}/\bar{I}$ to get $\overline{B}_1 \in \operatorname{End}(\overline{Q}_1)$ such that $\bar{h}\overline{B}_1$: $\overline{Q}_1 \twoheadrightarrow \bar{I}$ is surjective and $\det \overline{B}_1 \in \overline{A}$ has image d^{-1}. Lift $\overline{B}_1$ to $B_1 \in \operatorname{End}(Q_1)$, and let $B = B_1 \oplus 1_{Q_2} \in \operatorname{End}(Q)$. This satisfies the desired conditions. ∎

We will need (3)(c) of the next proposition in the proof of Theorem 6.1.

PROPOSITION 5.7. *Let B be a ring, F a free B-module of rank r, M a B-module, and g: $F \twoheadrightarrow M$ a surjection. Let $L = \operatorname{Ann}_B(\wedge^r M)$ and $I \subseteq B$ be any ideal. Then we have*:

(1) $\wedge^r (M/IM) \cong B/(I + L)$.

(2) *M/IM is B/I-free of rank r if and only if $I \supseteq L$.*

(3) *Suppose* $\theta \in \mathrm{GL}(F)$, $c \in B^\times$ *is a unit, and* $\det \theta \equiv c \bmod L$. *Suppose one of the following conditions holds.*

(a) *B is local and $L = B$,*

(b) *L is a nilpotent ideal, or*

(c) *B is Artinian.*

Then there exists $\theta' \in \mathrm{GL}(F)$ such that $g\theta = g\theta'$ and $\det \theta' = c$.

Proof. Since $\wedge^r g$: $B \cong \wedge^r F \twoheadrightarrow \wedge^r M$ is surjective, $\wedge^r M$ is a cyclic module, so $\wedge^r M \cong B/L$. Thus $\wedge^r M/IM \cong \wedge^r M \otimes_B B/I \cong B/L \otimes_B B/I \cong B/(I + L)$. So we have (1).

Claim. Let $x_1, \dots, x_r \in M$ be the image via g of a basis of F. Let $L_1 \subseteq B$ be generated by coefficients $a_i \in B$ which appear in equations $a_1 x_1 + \cdots + a_r x_r = 0$. Then $L = L_1$. Further, the map $F/LF \twoheadrightarrow M/LM$ induced by g is an isomorphism.

Proof. First we show $\bar{g}$: $F/L_1 F \twoheadrightarrow M/L_1 M$ is an isomorphism. Since the x_i are the image of a basis, it suffices to show that if $c_1 x_1 + \cdots + c_r x_r \in L_1 M$, then for all i, $c_i \in L_1$. But if $c_1 x_1 + \cdots + c_r x_r = d_1 x_1 + \cdots + d_r x_r$, with $d_i \in L_1$, then $(c_1 - d_1)x_1 + \cdots + (c_r - d_r)x_r = 0$, so by definition of L_1, $c_i - d_i \in L_1$. Thus $c_i \in L_1$ as desired.

Since $\bar{g}$ is an isomorphism, $\wedge^r M/L_1 M \cong \wedge^r F/L_1 F \cong B/L_1$, and so by (1), $L_1 = L_1 + L$, i.e., $L \subseteq L_1$. To see $L_1 \subseteq L$, suppose $b_1 x_1 + \cdots + b_r x_r = 0$. Fix i, $1 \le i \le r$. Then $b_i(x_1 \wedge \cdots \wedge x_r) = x_1 \wedge \cdots \wedge (\sum_{j \ne i} - b_j x_j) \wedge \cdots \wedge x_r = 0$, so b_i annihilates the generator $x_1 \wedge \cdots \wedge x_r$ of $\wedge^r M$, i.e., $b_i \in L$. Thus $L_1 \subseteq L$, and the claim is complete. ∎

Now we show (2). If M/IM is free of rank r, then $\wedge^r M/IM \cong B/I$, so by (1), $I = I + L$, i.e., $I \supseteq L$. Conversely, suppose $I \supseteq L$. By the above claim, $F/LF \xrightarrow{\sim} M/LM$ is an isomorphism, thus so is $F/IF \xrightarrow{\sim} M/IM$, which tells us that M/IM is rank r free.

For (3), notice that (3)(c) follows from the other cases, by expressing B as a product of Artinian local rings, and applying (3)(a) or (3)(b) to each factor.

To prove (3)(a), by the above claim, there is a linear dependence $a_1 x_1 + \cdots + a_r x_r = 0$ with some a_i a unit. By multiplying through by a_i^{-1} and reordering, we may assume that $a_1 = 1$. Let M be the following $r \times r$ matrix.

$$M = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_r & 0 & \cdots & 0 \end{pmatrix}.$$

Let $\theta' = (1_r + (c \det \theta^{-1} - 1)M)\theta$. Since M has image in $\ker g$, $g\theta = g\theta'$. Since $a_1 = 1$, $\det \theta' = (c \det \theta^{-1}) \det \theta = c$; since c is a unit, $\theta' \in \mathrm{GL}(F)$, as desired.

Finally, we show (3)(b). By the claim, let $a_{i1} x_1 + \cdots + a_{ir} x_r = 0$ be equations, $1 \le i \le m$, such that the a_{ij} generate the ideal $L_0 \subseteq L$, with $\det \theta - c \in L_0$. (We use the ideal L_0 since L may not be finitely generated.) We now inductively establish:

Claim. For any $k \ge 1$, there exists $\theta_k \in \mathrm{GL}(F)$ such that $g\theta_k = g\theta$ and $\det \theta_k \equiv c \bmod L_0^k$.

Proof. For $k = 1$, we select $\theta_1 = \theta$, of course. Assume the claim for some fixed $k \ge 1$. Let $d = \det \theta_k \in B^\times$. Since $d - c \in L_0^k$, we have $d - c = \sum_{i,j} b_{ij} a_{ij}$, for some $b_{ij} \in L_0^{k-1}$. For $1 \le i \le m$ and $1 \le j \le r$, let M_{ij} be the following $r \times r$ matrix, with the a_{i*} in the jth column,

$$M_{ij} = d^{-1} b_{ij} \begin{pmatrix} 0 & \cdots & 0 & a_{i1} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{ir} & 0 & \cdots & 0 \end{pmatrix}.$$

Let $\theta_{k+1} = (\prod_{i,j}(1_r - M_{ij}))\theta_k$. (The product over indices i, j may be taken in any order.) Since the image of each M_{ij} is in $\ker g$, $g\theta_{k+1} = g\theta_k = g\theta$. And $\det \theta_{k+1} = (\prod_{i,j}(1 - d^{-1}b_{ij}a_{ij}))d \equiv (1 - \sum_{i,j} d^{-1}b_{ij}a_{ij})d = d - \sum_{i,j} b_{ij}a_{ij} = c \bmod L_0^{k+1}$. Since L_0^{k+1} is nilpotent and $c \in B^\times$, $\det \theta_{k+1} \in B^\times$, so $\theta_{k+1} \in \mathrm{GL}(F)$. The claim is complete. ∎

For (3)(b), apply the latter claim for k large enough so that $L_0^k \subseteq L^k = 0$. ∎

6. SUBTRACTION

We need the following definition in order to state our subtraction results.

DEFINITION. For A a ring and $I \subseteq A$ an ideal, we say I is a *quadratic lifting ideal* if for any unit $d \in (A/I)^\times$, there exists a unit $u \in A^\times$ such that $ud \in A/I$ is a square (i.e., $ud = b^2$ for some $b \in A/I$).

THEOREM 6.1 (Subtraction Result). *Let A be a Noetherian ring of dimension $n \geq 3$. Let P and Q be projective A-modules of rank n, Q having a unimodular element, with $\wedge^n P \cong \wedge^n Q$. Let $\tilde{I}$ and I be comaximal ideals with* $\dim A/\tilde{I} \leq n - 2$. *Suppose $I = I_1 \cap I_2$ with I_1 and I_2 comaximal,* $\dim A/I_1 \leq 0$, *and* $\dim A/I_2 \leq 1$. *Suppose we have one of the following sets of conditions.*

(a) *I_2/I_2^2 is a projective A/I_2-module, and A is a finite type $\mathbb{Z}$-algebra.*

(b) *I_2/I_2^2 is a projective A/I_2-module of rank $\leq n - 1$, and $L = \mathrm{Ann}_A(\wedge^n I_1/I_1^2)$ is a quadratic lifting ideal.*

If there exist surjections $Q \twoheadrightarrow I$ and $P \twoheadrightarrow \tilde{I} \cap I$, then there exists a surjection $P \twoheadrightarrow \tilde{I}$.

After proving Theorem 6.1, we will discuss what sort of ideals are quadratic lifting ideals, and more generally we will discuss applications of Theorem 6.1. Note that by Theorem 3.5, we have a converse of this theorem, in the sense that if we have surjections $Q \twoheadrightarrow I$ and $P \twoheadrightarrow \tilde{I}$, then we can get a surjection $P \twoheadrightarrow \tilde{I} \cap I$.

We remark that in (b), we have $L = \mathrm{Ann}_A(\wedge^n I_1/I_1^2) = \mathrm{Ann}_A(\wedge^n I)$. Indeed, by Proposition 6.2 below, $\wedge^n I \cong \wedge^n I/I^2 \cong \wedge^n(I/I_1 I \oplus I/I_2 I) \cong \wedge^n(I_1/I_1^2 \oplus I_2/I_2^2) \cong \wedge^n I_1/I_1^2$. The last isomorphism exists because I_1 and I_2 are comaximal, and I_2/I_2^2 being projective of rank $\leq n - 1$ implies $\wedge^n I_2/I_2^2 = 0$.

PROPOSITION 6.2. *Let A be a ring, P a projective module of rank $r \geq 2$, and $P \twoheadrightarrow I$ a surjection onto an ideal $I \subseteq A$. Then $I \subseteq \operatorname{Ann}_A(\wedge^r I)$ and the natural map $\wedge^r I \to \wedge^r I/I^2$ is an isomorphism.*

Proof. Since we can check our conclusions locally, we may assume P is free. If $x_1, \dots, x_r \in I$ is the image of a basis of P, then $x_1 \wedge \cdots \wedge x_r$ generates $\wedge^r I$. For fixed $1 \leq i \leq r$, select $1 \leq j \leq r$ with $j \neq i$. Then $x_i(x_1 \wedge \cdots \wedge x_r) = x_j(x_1 \wedge \cdots \wedge x_{j-1} \wedge x_i \wedge x_{j+1} \wedge \cdots \wedge x_r) = 0$, so $x_i \in \operatorname{Ann}_A(\wedge^r I)$. Since the x_i generate I, I annihilates $\wedge^r I$. This also shows $\wedge^r I \cong \wedge^r I \otimes_A A/I \cong \wedge^r I/I^2$. ∎

Proof of Theorem 6.1. Fix a surjection $g: P \twoheadrightarrow \tilde{I} \cap I$. There exists a surjection $Q \twoheadrightarrow I$, and by Corollary 3.3, we may assume $Q = A^{n-1} \oplus \wedge^n P$. (We will be using Proposition 5.6, so we need Q to have a free direct summand of rank 2.) Select $\nu: \wedge^n Q \xrightarrow{\sim} \wedge^n P$ an isomorphism. Since $\dim A/I \leq 1$, by Corollary 2.5(2), select $\alpha: Q/IQ \xrightarrow{\sim} P/IP$ an isomorphism such that $\wedge^n \alpha = \nu \otimes_A A/I$. Let "bar" denote mod I, and consider the following diagram.

$$\begin{tikzcd}
Q/IQ \arrow[r, dashed, "\theta"] \arrow[d, dashed, "\bar{h}"'] & Q/IQ \arrow[r, "\alpha"] & P/IP \arrow[d, "\bar{g}"] \\
I/I^2 \arrow[rr, equal] & & \tilde{I} \cap I/I(\tilde{I} \cap I)
\end{tikzcd}$$

Claim. We can find a surjection $h: Q \twoheadrightarrow I$ and $\theta \in \mathrm{GL}(Q/IQ)$ which make the above diagram commute. In (b), we may also assume that $\det \theta \in A/I$ is a square.

Suppose the claim is established. Let $d = \det \theta \in (A/I)^{\times}$, which in (b) is a square. We apply Proposition 5.6 to find $B \in \operatorname{End}(Q)$ such that $hB: Q \twoheadrightarrow I$ is surjective and $\det \overline{B} = d^{-1}$. Replacing h by hB and θ by $\theta\overline{B}$, we may assume that $\det \theta = 1$. Then $\wedge^n(\alpha\theta) = \wedge^n \alpha = \nu \otimes_A A/I$ and the above diagram commutes. So we can now apply Theorem 4.5 to get the desired surjection $P \twoheadrightarrow \tilde{I}$.

All that remains is to prove the claim. Since $A/I \cong A/I_1 \times A/I_2$, proving the claim is equivalent to finding $h: Q \twoheadrightarrow I$ and $\theta_i \in \mathrm{GL}(Q/I_iQ)$, $i = 1, 2$, such that the following diagram commutes for $i = 1, 2$.

$$\begin{tikzcd}
Q/I_iQ \arrow[r, "\theta_i"] \arrow[d, "h \otimes_A A/I_i"'] & Q/I_iQ \arrow[rr, "\alpha \otimes_{A/I} A/I_i", "\cong"'] & & P/I_iP \arrow[d, "g \otimes_A A/I_i"] \\
I/I_iI \arrow[r, equal] & I_i/I_i^2 \arrow[rr, equal] & & \tilde{I} \cap I/I_i(\tilde{I} \cap I)
\end{tikzcd}$$

In (b), we must also have $\det \theta_i$ a square, $i = 1, 2$.

Fix $h\colon Q \twoheadrightarrow I$ any surjection. In (a), we can apply Proposition 5.4 to find θ_1 and Lemma 5.1 to find θ_2. So we need only now consider (b). By Proposition 5.4, find $\theta_1 \in \mathrm{GL}(Q/I_1Q)$ such that the above diagram commutes for $i = 1$. Since L (which contains I_1) is, by assumption, a quadratic lifting ideal, find $u \in A^\times$ such that $u \det \theta_1$ maps to a square in A/L. Since Q has a unimodular element, we can find $U \in \mathrm{GL}(Q)$ such that $\det U = u$. Replacing h by hU and θ_1 by $\theta_1(U \otimes_A A/I_1)$, we may assume that $\det \theta_1$ maps to a square of A/L. By applying Proposition 5.7(3)(c) to the ring $B = A/I_1$ and the surjection $(g \otimes_A A/I_1)(\alpha \otimes_{A/I} A/I_1)$, we may replace θ_1, so that $\det \theta_1$ is itself a square. (Note that Q/I_1Q is rank r free by Corollary 2.5(1).) Now apply Lemma 5.1 to find θ_2 making the above diagram commute, with $\det \theta_2 = 1$. This completes the claim. ∎

Remark 6.3. Note in the above proof that if I_1 is itself a quadratic lifting ideal, then we do not need to apply Proposition 5.7(3)(c). In all of the applications of Section 1 and in the applications below, we use this limited case where I_1 is itself a quadratic lifting ideal.

Now let us look at examples of quadratic lifting ideals. If A is an affine algebra over a field F and $\mathfrak{m}$ is the maximal ideal of an F-rational point, then $\mathfrak{m}$ is a quadratic lifting ideal, since $(A/\mathfrak{m})^\times \cong F^\times \subseteq A^\times$. If A is any ring and $I \subseteq A$ is an ideal such that A/I is quadratically closed (i.e., every element is a square), then I is a quadratic lifting ideal. For example, I could be the intersection of a finite number of maximal ideals $\mathfrak{m}_i$ such that for all i, $A/\mathfrak{m}_i$ is algebraically closed (or just quadratically closed). Combining these ideas, suppose A is an affine algebra over a field F, $\mathfrak{m}_0$ is a maximal ideal of an F-rational point, and $\mathfrak{m}_1, \dots, \mathfrak{m}_t$ are maximal ideals such that $A/\mathfrak{m}_i$ is quadratically closed for $1 \le i \le t$. Then $J = \mathfrak{m}_0 \cap \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_t$ is a quadratic lifting ideal. (Indeed, given $d \in (A/J)^\times$, $d = (d_0, d_1, \dots, d_t)$, $d_i \in A/\mathfrak{m}_i$, we can find $u \in A^\times$ such that $ud = (1, d_1', \dots, d_t')$, which is a square.) We can say more, however. If $\operatorname{char} F \neq 2$, $J = \mathfrak{m}_0 \cap \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_t$ is as above, and I is an ideal such that $\sqrt{I} = J$, then I is a quadratic lifting ideal by the following proposition.

PROPOSITION 6.4. *Let A be a ring and $I \subseteq J \subseteq A$ ideals. If A/I is Artinian and I is a quadratic lifting ideal, then J is a quadratic lifting ideal. Conversely, if A/I is Noetherian, J is a quadratic lifting ideal, $\sqrt{I} = \sqrt{J}$, and* 2 *is a unit of A/I, then I is a quadratic lifting ideal.*

Proof. The first statement follows directly from the definition of a quadratic lifting ideal since every unit of A/J is the image of a unit of A/I. For the second statement, we need the following lemma.

LEMMA 6.5. *Let B be any ring, $J \subseteq B$ a nilpotent ideal. Fix an integer $r \geq 1$ and suppose r is a unit in B. Let $a \in B^\times$ be a unit, and suppose $a \equiv b_1^r \bmod J$. Then $a = b^r$ for some $b \in B$.*

Proof. Since J is nilpotent, find N such that $J^N = 0$. We induct on N, $N = 1$ being trivial. For $N > 1$, by applying our induction hypothesis to the ring B/J^{N-1}, we can find $b_2 \in B$ such that $a \equiv b_2^r \bmod J^{N-1}$. Since $J \subseteq \operatorname{rad} B$ and a is a unit, b_2 is a unit. Let $b = b_2 + (rb_2^{r-1})^{-1}(a - b_2^r)$. Since $(J^{N-1})^2 = 0$, $b^r = a$, as desired. ∎

To prove the other statement of Proposition 6.4, let $d \in (A/I)^\times$ be a unit. Then there exists $u \in A^\times$ such that the image of ud in A/J is a square. Applying Lemma 6.5 with $B = A/I$, $J = J/I$, and $a = ud$, we see that $ud \in A/I$ is itself a square. This completes our proposition. ∎

In looking at where Theorem 6.1 can be applied, let us begin with some general remarks. Any of $\tilde{I}$, I_1, or I_2 may be equal to A. (In particular, $\tilde{I} = A$ will give as a conclusion that P has a unimodular element.) We also remark that if I/I^2 is a projective A/I-module, then automatically I_2/I_2^2 is a projective A/I_2-module. Finally, note that in (b), since $L \supseteq I_1$, we know by Proposition 6.4 that if I_1 is a quadratic lifting ideal, then so is L. (Also see Remark 6.3.) Now let us compare Theorem 6.1 to the subtraction results of [MS], which we recall from Section 1.

THEOREM. *Let A be a Noetherian ring of dimension $n \geq 3$ and P be a projective A-module of rank n with trivial determinant. Let $\tilde{I}$ and I be comaximal ideals of A of height n. Suppose we have surjections $A^n \twoheadrightarrow I$ and $P \twoheadrightarrow \tilde{I} \cap I$. Then there exists a surjection $P \twoheadrightarrow \tilde{I}$ provided either of the following sets of conditions holds.*

(1) [MS, *Theorem* 3.5] *A is an affine algebra over a field F and I is the maximal ideal of an F-rational point.*

(2) [MS, *Theorem* 3.14] *I/I^2 is a free A/I-module of rank n and one of the following conditions holds.*

(i) *A is a finitely generated $\mathbb{Z}$-algebra.*

(ii) *A is an affine algebra over a field F and A/I is the product of quadratically closed fields.*

(iii) *A is an affine algebra over a field F,* $\operatorname{char} F \neq 2$, *and $A/\sqrt{I}$ is a product of quadratically closed fields.*

First we note that in Theorem 6.1, we weaken the hypothesis $\operatorname{ht} \tilde{I} = n$ to $\dim A/\tilde{I} \leq n-2$; we made the same extension for addition in Theorem 3.5. Also as for addition, we allow P to have non-trivial determinant by replacing A^n with Q, a projective module with a unimodular element, of equal rank and determinant to P.

Let us consider what ideals we can subtract in the case where $I = I_2$, i.e., $I_1 = A$. That is, I can be any ideal with $\dim A/I \leq 1$ such that I/I^2 is a projective A/I-module of rank $\leq n - 1$. (In the case of a $\mathbb{Z}$-algebra, we can drop the restriction on the rank of I/I^2.) This covers, for example, any local complete intersection ideal of height $n - 1$. It also covers an intersection of finitely many maximal ideals $\mathfrak{m}_i$ of height $\leq n - 1$ which define regular points. All of these examples are completely new, as [MS] only looked at I having height n.

Next, let us look at the case where we have $I = I_1$, i.e., $I_2 = A$. These examples will be generalizations of results of [MS]. The first observation is that in [MS, Theorem 3.14], it turns out that requiring I/I^2 to be a rank n free A/I-module is entirely unnecessary. So, for instance, when we are looking at a finitely generated $\mathbb{Z}$-algebra, we can now subtract any ideal I with $\dim A/I = 0$. In applications, requiring I/I^2 to be rank n free is fairly restrictive. For example, if I is a finite intersection of maximal ideals, I/I^2 is rank n free if and only if every maximal ideal defines a regular point.

Further, in the case of an affine algebra over a field F [resp. over a field F with $\operatorname{char} F \neq 2$], notice that by our discussion above, we can subtract any ideal which defines [resp. whose radical defines] the union of a rational point with closed points whose residue fields are quadratically closed. In particular, if $\operatorname{char} F \neq 2$, we can subtract off an ideal whose radical defines a rational point. As pointed out in Section 1, this particular result has been used by Bhatwadekar and Sridharan to produce certain counterexamples [BS].

Finally, we can combine the cases of $I = I_1$ and $I = I_2$ by taking the general case where $I = I_1 \cap I_2$ with I_1 and I_2 comaximal, and having forms as described above. This is a non-trivial extension, just as in the case of combining a rational point with points with quadratically closed residue fields. This is because we are not able to simply iteratively subtract off first I_1 and then I_2, because Theorem 6.1 requires having $Q \twoheadrightarrow I$, and we may have $Q \twoheadrightarrow I_1 \cap I_2$ without having both $Q' \twoheadrightarrow I_1$ and $Q'' \twoheadrightarrow I_2$.

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